Algorithms for Hierarchical and Semi-Partitioned Parallel Scheduling

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Motivation: Scheduling jobs on computing clusters

Given: a set of computational jobs

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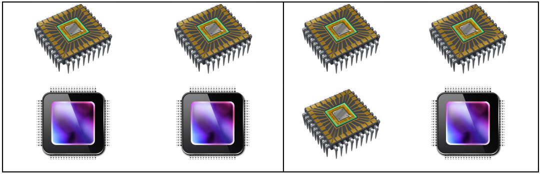

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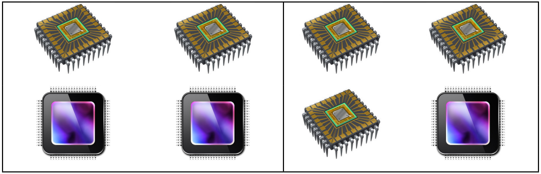

Two types of approach: with or without migration

1. Partitioned: Confine each job to a specific machine
2. Global: Allow jobs to migrate between machines (and clusters)

Partitioned scheduling vs. global scheduling

Partitioned scheduling

+ No migration overheads
– More constrained, smaller set of schedulable instances

Global scheduling

+ Less constrained, larger set of schedulable instances
– Migration overheads

Hybrid approaches

- Semi-partitioned scheduling
    - pre-assign some of the jobs to the machines
    - allow global migrations for the rest
- Clustered scheduling:
    - pre-assign each job to a machine cluster
    - allow migrations inside each cluster

Bastoni, Brandenburg & Anderson (2010):
experimental comparison of the trade-offs

Processor affinity

- System interface to restrict the set of processors on which a job may be scheduled
- Widely available across operating systems:
  - Linux: `sched_setaffinity()`
  - FreeBSD: `cpuset_setaffinity()`
  - Windows: `SetThreadAffinityMask()`

Example of affinity mask settings

	mach. 1	mach. 2	mach. 3	mach. 4
job 1	x	x	–	–
job 2	x	x	x	x
job 3	–	–	x	x
job 4	x	–	–	–

Example of affinity mask settings

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job 1	x	x	–	–
job 2	x	x	x	x
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job 4	x	–	–	–

- Question: How to set affinity masks to achieve good tradeoffs? And how to model the tradeoffs in the first place?

Our formalization

Idea: allow the processing time to depend on the affinity mask of the job

- jobs $J = \{1, \dots, n\}$
- machines $M = \{1, \dots, m\}$
- a family of admissibile sets $\mathcal{A} \subseteq 2^M$ (the available masks)
- for each $j \in J$, a function $P_j : \mathcal{A} \to \mathbb{Z}_+$
  - monotone: $\alpha \subset \beta \Rightarrow P_j(\alpha) \leq P_j(\beta)$

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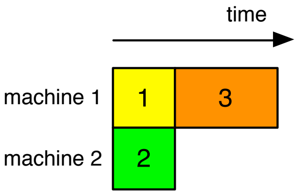
Goal: for each job j, find a set $\bar{\alpha}_j \in \mathcal{A}$ and a schedule of j on $\bar{\alpha}_j$ that minimizes the makespan

An example

$J = \{1, 2, 3\}$
$M = \{1, 2\}$
$\mathcal{A} = \{\{1\}, \{2\}, \{1, 2\}\}$

j	$P_j(\{1\})$	$P_j(\{2\})$	$P_j(\{1,2\})$
1	4	∞	∞
2	∞	4	∞
3	7	7	10

A possible solution:



makespan = 11

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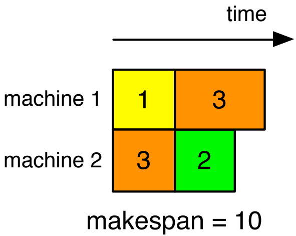

Note: simultaneous parallel processing of the same job is not allowed

Examples with different $\mathcal{A}$

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In many cases, the family $\mathcal{A}$ is hierarchical, or laminar:

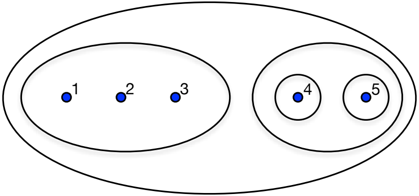
- $\alpha, \beta \in \mathcal{A} \Rightarrow \alpha \subseteq \beta \lor \beta \subseteq \alpha \lor \alpha \cap \beta = \emptyset$

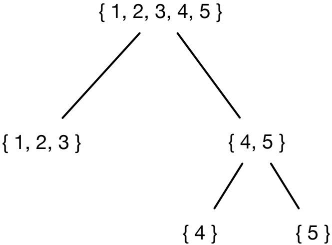
We will assume $\mathcal{A}$ laminar for our results
(the model makes sense even without this assumption)

We call the resulting problem HIERARCHICAL SCHEDULING

Structure of laminar families

Say $\mathcal{A} = \{\{1,2,3\},\{4\},\{5\},\{4,5\},\{1,2,3,4,5\}\}$





There are never "too many" affinity masks: $|\mathcal{A}| \leq 2m$

Hardness of Hierarchical Scheduling

Hierarchical Scheduling generalizes the $R||C_{\max}$ problem (scheduling on unrelated machines)

$\Rightarrow$ Computing solutions of makespan less than $\left(\frac{3}{2} - \epsilon\right) \cdot \mathrm{opt}$ is NP-hard (Lenstra, Shmoys & Tardos 1987)

opt: minimum makespan of a solution
ϵ: arbitrary positive constant

Approximability: our result

Let $\rho \geq 1$. A ρ-approximation algorithm outputs, in polynomial time given any HS instance I, a solution with makespan $\leq \rho \cdot \mathrm{opt}(I)$

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Main Result (B., D'Angelo, Marchetti-Spaccamela)

Hierarchical Scheduling admits a 2-approximation algorithm.

Related work on unrelated machines

- Lenstra, Shmoys & Tardos (1987): $R||C_{\max}$ admits a 2 approximation

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- Lenstra, Shmoys & Tardos (1987): $R||C_{\max}$ admits a 2 approximation
  Nothing better than a $2 - \frac{1}{m}$ is known, even today

Outline of the approach

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2. Prove the ILP formulation is exact
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Outline of the approach

1. Integer Linear Programming formulation of the problem
2. Prove the ILP formulation is exact
    - The ILP constraints are necessary (trivial)
    - Show that given ILP solution $(\mathbf{x}, T)$, one can construct a valid schedule of makespan T using the affinity masks described by $\mathbf{x}$
3. Show how to approximately round the ILP
    - Leverage the LP structure to redistribute the fractional values on the leaves of $\mathcal{A}$
    - Invoke Lenstra-Shmoys-Tardos rounding to get solution $(\bar{\mathbf{x}}, 2T)$ with $\bar{\mathbf{x}}$ integral

ILP formulation for Semi-Partitioned Scheduling

p_{ij}: shorthand for $P_j(\{i\})$
p_{0j}: shorthand for $P_j(M)$
x_{ij}: 1 if j assigned to machine i, 0 otherwise
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$$\min \; T \tag{IP-1}$$

$$\sum_{i=0}^{m} x_{ij} = 1 \qquad \text{for } j = 1, \ldots, n \tag{1}$$

$$\sum_{j=1}^{n} p_{ij} x_{ij} \leq T \qquad \text{for } i = 1, \ldots, m \tag{2}$$

$$\sum_{j=1}^{n} \sum_{i=0}^{m} p_{ij} x_{ij} \leq mT \tag{3}$$

$$p_{ij} x_{ij} \leq T \qquad \text{for } j = 1, \ldots, n \text{ and } i = 0, \ldots, m \tag{4}$$

ILP formulation for Hierarchical Scheduling

$p_{\alpha j}$: shorthand for $P_j(\alpha)$
$x_{\alpha j}$: 1 if j assigned affinity mask α, 0 otherwise

$$\min \; T \tag{IP-2}$$

$$\sum_{\alpha \in \mathcal{A}} x_{\alpha j} = 1 \qquad \text{for } j = 1, \ldots, n \tag{5}$$

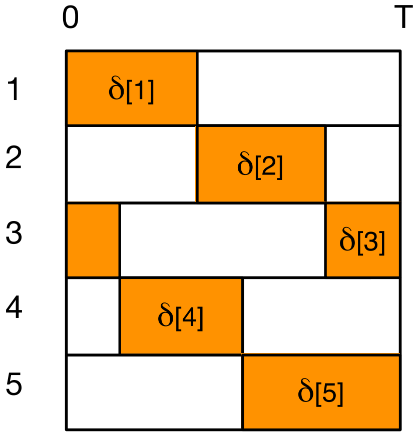
$$\sum_{j=1}^{n} \sum_{\beta \subseteq \alpha} p_{\beta j} x_{\beta j} \leq |\alpha| T \qquad \text{for each } \alpha \in \mathcal{A} \tag{6}$$

$$p_{\alpha j} x_{\alpha j} \leq T \qquad \text{for each } \alpha \in \mathcal{A} \text{ and } j = 1, \ldots, n \tag{7}$$

Constructing the schedule: Semi-Partitioned case

$t \leftarrow 0$; $i \leftarrow 0$;
$V \leftarrow \sum_{j=1}^{n} p_{0j} x_{0j}$;
while $V > 0$ **do**
  $i \leftarrow i + 1$;
  $\delta \leftarrow \min(V, T - \sum_{j=1}^{n} p_{ij} x_{ij})$;
  Assign δ units of global jobs to i, in the interval $[t, t + \delta \pmod{T})$];
  $t \leftarrow t + \delta \pmod{T}$;
  $V \leftarrow V - \delta$;
foreach machine $i \in M$ and job $j \in J$ such that $x_{ij} = 1$ **do**
  Schedule j on machine i in the idle time of interval $[0, T]$;

Example



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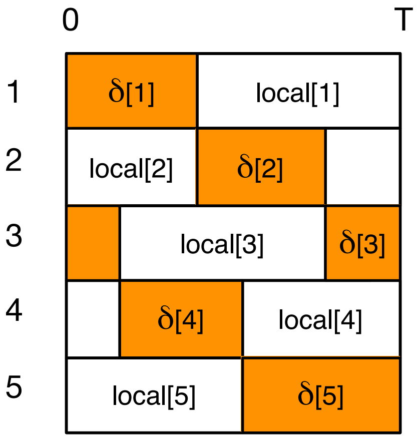

Constructing the schedule: Hierarchical case (sketch)

1. Bottom-up volume allocation phase:
   - Traverse $\mathcal{A}$ from local to global masks
   - Compute a "share" $\text{LOAD}[i, \alpha]$ of the jobs with mask α on machine i
   - Greedily assign the shares to more restricted machines first

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2. Top-down job assignment phase:
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Lemma

If $(\mathbf{x}, T)$ feasible for ILP, there exists a valid schedule with makespan T.

(In particular, no job is simultaneously scheduled on distinct machines)

Preprocessing the ILP

$p_{\alpha j}$: shorthand for $P_j(\alpha)$
$x_{\alpha j}$: 1 if j assigned affinity mask α, 0 otherwise

$$
\begin{aligned}
& \cancel{\min\ T} && && \text{(IP-2)} \\
& \sum_{\alpha \in \mathcal{A}} x_{\alpha j} = 1 && \text{for } j = 1, \dots, n && \\
& \sum_{j=1}^{n} \sum_{\beta \subseteq \alpha} p_{\beta j} x_{\beta j} \le |\alpha| T && \text{for each } \alpha \in \mathcal{A} && \\
& \cancel{p_{\alpha j} x_{\alpha j} \le T} && \cancel{\text{for each } \alpha \in \mathcal{A} \text{ and } j = 1, \dots, n} &&
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Binary search for T
Remove the $x_{\alpha j}$ with $p_{\alpha j} > T$
Decide either: (a) target T is infeasible, or (b) $2T$ is feasible

Fractional values “push-down” lemma

Lemma

Let $\mathbf{x}$ be LP-feasible, let $\beta \in \mathcal{A}$ ($|\beta| > 1$).
There exists another LP-feasible solution $\mathbf{x}'$ such that, for each job j,

$$x'_{\alpha j} = x_{\alpha j} \qquad \text{whenever } \alpha \not\subseteq \beta, \text{ and}$$
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$\Rightarrow$ Invoke any LP-based rounding for $R||C_{\max}$ to obtain 2-approximation

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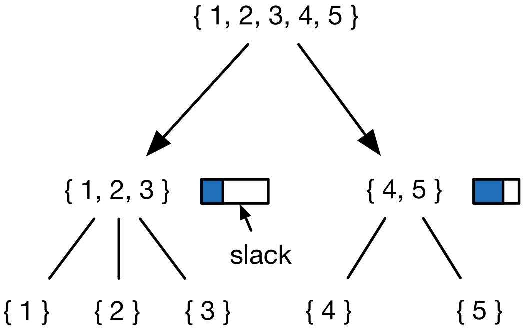

Intuition: distribute value to children proportionally to their "slack":

$$\mathrm{slack}(\alpha) \stackrel{\mathrm{def}}{=} |\alpha| \cdot T - \sum_{j \in J} \sum_{\beta \subseteq \alpha} p_{\beta j} x_{\beta j}.$$

Summary and open problems

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